Evaluation of two improper integrals via Rayleigh energy theorem

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1. Fascinating world of improper integrals

Inversions in mathematics are typically more difficult than forward operations. There are many examples in arithmetic, linear algebra and calculus – division vs multiplication (particularly for polynomials and matrices), powers vs roots, integrals vs derivatives.

In calculus, there are many cases where antiderivatives (indefinite integrals) of innocent-looking functions cannot be expressed in terms of elementary functions. In fact, a randomly chosen non-polynomial function is likely to not have an elementary antiderivative. Yet, every so often, when we attempt to evaluate such an integral within limits, beautiful closed-form solutions pop out. When one or both limits these integrals approach ±∞ the integrals are known as improper integrals.

Improper integrals are not merely theoretical curiosities, but also routinely arise in practical scientific and engineering situations. I’ll quote a few examples below.

Gaussian integral: $\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$ (1)

Sinc integral: $\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx = \pi$ (2)

Fresnel integral: $\int_{-\infty}^{\infty} \sin(x^2) \, dx = \sqrt{\frac{\pi}{2}}$ (3)

Planck’s law integral: $\int_{0}^{\infty} \frac{x^3}{e^x - 1} \, dx = \frac{\pi^4}{15}$ (4)

A full frontal attack on these integrals is seldom successful. These integrals are usually evaluated using one of the two techniques: (1) the residue theorem or (2) differentiation under the integral sign. The former technique requires a long journey through complex analysis and the latter requires discovery of auxiliary functions with the right parameters that essentially cancel out the difficult parts of the function (e.g. the $x$ in the denominator of $\frac{\sin x}{x}$).

Right from the time I first encountered improper integrals, I’ve been fascinated by them. In my free time, I often try to find ways to evaluate them using elementary means. Of course, terms like “elementary” and “intuitive” are subjective and depend on one’s familiarity with the subject. To some, methods of complex analysis may be intuitive and differentiation under integral sign may feel like trickery. Regardless, this post is about evaluation of two improper integrals (Eqs. (1) and (2)) using Rayleigh energy theorem from Fourier analysis. It may be possible to use this method to evaluate more such integrals. Below, I’ll first state my convention for Fourier transform pairs followed by the Rayleigh energy theorem (without proof).

2. Fourier transform and the Rayleigh energy theorem

Fourier transform (and Fourier Series, its discrete counterpart) are techniques for decomposing a signal into its frequencies and the reverse operation of reconstructing a signal given its frequency components. The signal can be spatial or temporal. For any given signal, $f(t)$, the
forward and reverse (transforms are expressed as:

\[ F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \quad \text{(Extracts spectrum, given signal)} \] (5)

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} d\omega \quad \text{(Extracts signal, given spectrum)} \] (6)

The above equations, called Fourier transform pair, express a remarkable fact: that a signal is completely determined by its frequency components (a.k.a spectrum). Both contain the same information and are derivable from each other. Engineering courses unfortunately often present these equations and teach us to manipulate them without communicating the full context of such a beautiful idea.

The Rayleigh energy theorem takes this equivalence even further. It states that the energy content of a signal can be computed either using the signal itself or by summing up the energy of the individual frequency components making up the signal. The answer you get from both these approaches should match up. In fact, RET’s statement is slightly more general than that. It defines the energy of a signal as the “overlap integral” of the signal with itself and states that the overlap of two signals or their frequency components is same. Mathematically

\[ \int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega \] (7)

\[ \Rightarrow \int_{-\infty}^{\infty} \|f(t)\|^2 dt = \int_{-\infty}^{\infty} \|F(\omega)\|^2 d\omega \] (8)

Eq. (8) has many practical applications in physics and engineering. But we’re about to use it to prove some pretty mathematical results.

3. Gaussian integral

The Gaussian function, colloquially called ‘the bell curve’, is one of the most important functions in Mathematics. It describes the distribution of many naturally occurring processes that cluster around some average value. As such, it is encountered in almost all branches of science ranging from statistics and physics to cognitive sciences. In its simplest form, the Gaussian function can be written as

\[ g(t) = e^{-t^2} \] (9)

The total area under this curve is need to “normalize” the various probability distribution. Like the function, the integral too is encountered in diverse branches of natural and applied sciences where cumulative statistics are desired. The Gaussian integral is stated as

\[ \mathcal{G} = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi} \] (10)

The conventional method of calculating the Gaussian integral, involving transformation to polar co-ordinates, is too well known to be repeated here. Instead we’ll use Rayleigh energy theorem (Eq. (8)) to prove Eq. (10).

Let us begin by defining

\[ f(t) = e^{-\frac{t^2}{2}} \] (11)

The Fourier transform of \( f(t) \), using the unitary definition of Fourier transform (Eq. (5)), can
be calculated as

\[ F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{-i\omega t} dt \]

\[ = e^{-\frac{\omega^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{(t+i\omega)^2}{2}} dt \]  \hspace{1cm} (12)

Now let \( t + i\omega = \sqrt{2}u; \ dt = \sqrt{2} du \) to rewrite Eq. (12) as

\[ F(\omega) = e^{-\frac{\omega^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} du \]

\[ = e^{-\frac{\omega^2}{2}} \frac{1}{\sqrt{\pi}} G \] \hspace{1cm} (13)

where we have used the definition of \( G \) from Eq. (10).

We now use the Rayleigh energy theorem, Eq. (8):

\[ \int_{-\infty}^{\infty} \left\| e^{-\frac{t^2}{2}} \right\|^2 dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \left\| e^{-\frac{u^2}{2}} \right\|^2 d\omega \]

\[ \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{G^2}{\pi} \int_{-\infty}^{\infty} e^{-\omega^2} d\omega \] \hspace{1cm} (14)

Because \( t \) and \( \omega \) are dummy variables, the integrals on the left and the right hand sides are both equal to \( G \). We therefore get the simple relation,

\[ G = \frac{1}{\pi} G^3 \] \hspace{1cm} (15)

from which we immediately obtain the Gaussian integral:

\[ G = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi} \] \hspace{1cm} (16)

4. Sinc function integral

The second function we consider is the sinc function

\[ f(t) = \frac{\sin t}{t} \] \hspace{1cm} (17)

It is not possible to express this the antiderivative of this function in terms of elementary functions. However the definite (improper) integral of this function exists and equals \( \pi \).

\[ \mathcal{A}_1 = \int_{-\infty}^{\infty} \frac{\sin t}{t} dt = \pi \] \hspace{1cm} (18)

Eq. (18) can once again be proved using residue integration or differentiation under integral sign. So far I’ve been unsuccessful in finding any other technique for a direct evaluation of this integral. I was, however, able to evaluate this using an indirect method as follows. We first evaluate a related integral

\[ \mathcal{A}_2 = \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} dt = \pi \] \hspace{1cm} (19)
using Rayleigh energy theorem and then show that $A_2 = A_1$.

Consider a box function defined on the real axis.

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & -\infty < t < 0 \text{ and } 1 < t < \infty \end{cases}$$

Its Fourier transform, using Eq. (5) is

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{1} e^{-i\omega t} \, dt$$

$$= \frac{1 - e^{-i\omega}}{i\omega}$$

$$= \sqrt{\frac{2}{\pi}} \sin(\omega/2) e^{-i\omega/2}$$

We now use the Rayleigh energy theorem to obtain

$$\int_{-\infty}^{\infty} F(\omega)^2 \, d\omega = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

$$\int_{-\infty}^{\infty} \left( \sqrt{\frac{2}{\pi}} \frac{\sin(\omega/2)}{\omega} \right)^2 e^{-i\omega/2} \, d\omega = 1$$

$$\int_{-\infty}^{\infty} \frac{2 \sin^2(\omega/2)}{\omega^2} \, d\omega = 1$$

We use a change of variables $u = \omega/2$, $d\omega = 2\,du$ we obtain Eq. (19).

Next we want to prove $A_1 = A_2$. It is provable directly. Consider

$$A_2 = \int_{-\infty}^{\infty} \sin^2 t \, dt = \int_{-\infty}^{\infty} \frac{1}{2} \cdot \frac{\sin^2 t}{t^2} \, dt$$

Let $u = \sin^2 t / t^2$ and $v' = 1 \Rightarrow v = t$. We use integration by parts

$$\int uv' \, dx = uv - \int vu' \, dx$$

to get

$$A_2 = \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} \, dt = \int_{-\infty}^{\infty} \frac{1}{2} \cdot \frac{\sin^2 t}{t^2} \, dt$$

$$= \frac{\sin^2 t}{t} \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} 2 \sin t \left( \frac{\cos t}{t} - \frac{\sin t}{t^2} \right) \, dt$$

Expand out the brackets and use $\sin 2t = 2 \sin t \cos t$ to obtain

$$A_2 = \int_{-\infty}^{\infty} \frac{2 \sin^2 t}{t^2} - \int_{-\infty}^{\infty} \frac{\sin 2t}{2t} \, d(2t)$$

$$= 2A_2 - A_1$$

$$\Rightarrow A_2 = A_1$$

Thus we obtain Eq. (2) without the use of residue integration or differentiation under integral sign.